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# Computation of tendencies and vertical motion with a two-parameter model of the atmosphere

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SUMMARY

The system of partial differential equations given by Sawyer and Bushby (1953) for the rates of change of the 1,000-500-mb thickness and the 500-mb contour height have been solved on an electronic computing machine for three synoptic situations and the fields of vertical motion were also computed. The results, described in this paper, show reasonable agreement with actuality. The implied 1,000-mb height tendencies also agree quite well with those actually observed.

## 1. INTRODUCTION

In recent years, several attempts have been made to forecast by numerical methods the large-scale atmospheric motions. Charney (1949) justified the use of the geostrophic approximation in evaluating the horizontal velocity and the vertical component of vorticity after the horizontal divergence has been eliminated from the basic equations. Charney also showed that if one assumes that the wind is uni-directional with respect to height and that the increase of wind with height is the same along all verticals, then there is one level in the atmosphere (approximately in mid-troposphere) at which the flow corresponds to the flow in a barotropic fluid. Several writers, including Charney, Fjörtoft and von Neumann (1950) and Bolin and Charney (1951), have tested this theory by applying the vorticity equation relating to the motion of a barotropic fluid to the 500-mb isobaric surface in order to predict changes in the height of that surface. The degree of success achieved with this model of the atmosphere was not sufficient to justify using it to produce routine forecasts, but was enough to warrant extending the model to take account of some of the baroclinic properties of the atmosphere.

Phillips (1951) indicated a method of representing some of the simple baroclinic properties of the atmosphere by considering two barotropic fluids superimposed one upon the other, the whole being contained between rigid horizontal plates. Charney and Phillips (1953) gave a generalization of this in which the atmosphere is divided into $n$ layers, the motion being assumed barotropic in each layer.

An alternative way in which some of the baroclinic features of the atmosphere can be incorporated in a simple model is to assume that the variation of the horizontal and vertical components of velocity with height is the same along all verticals. Eady (1952) gave the title of '$2\frac{1}{2}$-dimensional model' to this type of model atmosphere, and Eliassen (1952), Thompson (1953) and Sawyer and Bushby (1953) have derived sets of equations which describe the motion in particular versions of this model.

Sawyer and Bushby derived, subject to various approximations, a set of simultaneous partial differential equations for forecasting the contour height, $h_m$, of an isobaric surface representative of mid-troposphere and the thickness, $h'$, of the layer from 1,000 mb to that isobaric surface. The method is one in which the instantaneous rates of change of $h_m$ and $h'$ are first computed at time $t$, and from these, new values of $h_m$ and $h'$ are computed for time $t + \delta t$. The process is then repeated $n$ times, and forecast charts are produced for time $t + n\delta t$.

The construction of a programme for the solution of this set of equations by an electronic computing machine is a formidable task. It was therefore decided to prepare one programme for the solution of the equation for the rate of change of contour height and another for the solution of the equation for the rate of change of thickness before proceeding with the step-by-step process of time integration. The present paper describes the computation of the rates of change of $h_m$ and $h'$ for three synoptic situations, and the computed results are compared with the actual changes that occurred. The effect of assuming wrong boundary conditions is discussed, as is the effect of neglecting the vertical advection of vorticity.

## 2. The Sawyer-Bushby $2\frac{1}{2}$-dimensional model

The model of the atmosphere adopted by Sawyer and Bushby (1953) consists of a baroclinic fluid in which the thermal wind is constant in direction in any vertical column, but not necessarily parallel to the wind direction at any level, and the thermal wind speed is assumed proportional to the pressure difference through the layer concerned. The atmosphere is considered bounded by two pressure surfaces across which the movement of air is assumed to be negligible. These surfaces are $p = p_1$ corresponding to the upper limit of the troposphere and $p = p_0$ corresponding to a level near the surface. Furthermore, the variation of vertical motion with respect to pressure is assumed to be parabolic. The geostrophic approximation is made after the horizontal divergence has been eliminated from the equations.

Conformally transforming the northern hemisphere by a stereographic projection from the south pole on to a plane perpendicular to the earth's axis and passing through the north pole, the following equations were derived for determining $\partial h_m/\partial t$, $\partial h'/\partial t$ and $\Pi$ where $\Pi = dp/dt$ is a measure of the vertical motion :

$$\nabla^2 \frac{\partial h'}{\partial t} + J(h_m, \beta^2 g l^{-1} \nabla^2 h') + J(h', \beta^2 g l^{-1} \nabla^2 h_m + l) = \frac{4\Pi_m (\nabla^2 h_m + l^2/\beta^2 g)}{(p_0 - p_1)} \quad . \quad (1)$$

$$\nabla^2 \frac{\partial h_m}{\partial t} + J(h_m, \beta^2 g l^{-1} \nabla^2 h_m + l) + \tfrac{1}{3} J(h', \beta^2 g l^{-1} \nabla^2 h') - \frac{8\Pi_m \nabla^2 h'}{3(p_0 - p_1)} = 0 \quad . \quad . \quad (2)$$

$$\frac{\partial h'}{\partial t} = \beta^2 g l^{-1} J(h', h_m) - RA\, \Gamma_p \Pi_m / g \quad . \quad . \quad . \quad (3)$$

where

$$A \equiv \tfrac{1}{2} \frac{p_0 + 3p_1}{p_0 - p_1} - \frac{4p_0 p_1}{(p_0 - p_1)^2} \log_e \frac{2p_0}{p_0 + p_1} \, ;$$

$\nabla^2$ and $J$ are the two-dimensional Laplacian and Jacobian operators referred to rectangular Cartesian coordinates on the plane projection of the earth's surface; $\beta$ is the magnification factor $\sec^2(\pi/4 - \theta/2)$, $\theta$ being the geographical latitude; $g$ is the acceleration due to gravity; $l$ is the Coriolis parameter; $\Pi_m$ is the value of $\Pi$ at the pressure level $p_m$, the mean level of the layer considered; $R$ is the universal gas constant and $\Gamma_p$ a measure of the departure of the lapse rate from the adiabatic given by $\Gamma_p = \partial T/\partial p - \gamma/g\rho$, $\gamma$ being the appropriate wet- or dry-adiabatic lapse rate, $\rho$ the density and $T$ the absolute temperature.

In the present work the values of $p_0$ and $p_1$ were taken to be 1,000 mb, and 200 mb respectively. This would give $p_m = 600$ mb, a level for which charts are not normally available. $h_m$ was therefore identified with the height of the 500 mb surface and $h'$ with the thickness of the 1,000-500 mb layer.

## 3. Method of solution

Eliminating $\Pi_m$ from Eqs. (1) and (3) gives

$$\nabla^2 \frac{\partial h'}{\partial t} + \frac{4g\,(\nabla^2 h_m + l^2/\beta^2 g)\{\beta^2 g\, l^{-1} J\,(h_m, h') + \partial h'/\partial t\}}{RA\,\Gamma_p\,(p_0 - p_1)}$$
$$+ J\,(h_m, \beta^2 g l^{-1} \nabla^2 h') + J\,(h', \beta^2 g l^{-1} \nabla^2 h_m + l) = 0 \qquad (4)$$

a Helmholtz type of differential equation in $\partial h'/\partial t$. Eqs. (4), (3) and (2) can be solved numerically to give values of $\partial h'/\partial t$, $\Pi_m$ and $\partial h_m/\partial t$ at a finite number of points, given the values of $\partial h'/\partial t$ and $\partial h_m/\partial t$ along the boundary of the area for which the solution is carried out. The following finite-difference approximations were used in evaluating the Jacobian and Laplacian operators :

$$(\nabla^2 z)_0 = \frac{1}{a^2}\left[\sum_1^4 z_1 - 4z_0\right] \qquad (5)$$

and

$$[J\,(x, y)]_0 = \frac{1}{4a^2}\,(x_1 - x_3)\,(y_2 - y_4) - (x_2 - x_4)\,(y_1 - y_3) \qquad (6)$$

where the suffixes refer to the grid points shown in Fig. 1 and where $a$ is the grid length. The grid length used in the present computations was about 162 statute miles in middle latitudes and a grid of $16 \times 12$ points was used. This enables values of $\partial h'/\partial t$ and $\partial h_m/\partial t$ to be computed at $12 \times 8$ points (excluding boundary points) as two rings of points are 'lost' in calculating the third-order differences necessary to evaluate the vorticity advection terms.

Eqs. (2) and (4) can be solved by the Liebmann iterative process. As Eq. (4) is a Helmholtz type of equation its evaluation converges more rapidly than the solution of Eq. (2), a Poisson type of equation. Twenty iterations were necessary to obtain the solution of Eq. (4) to the required degree of accuracy whereas 40 iterations were necessary

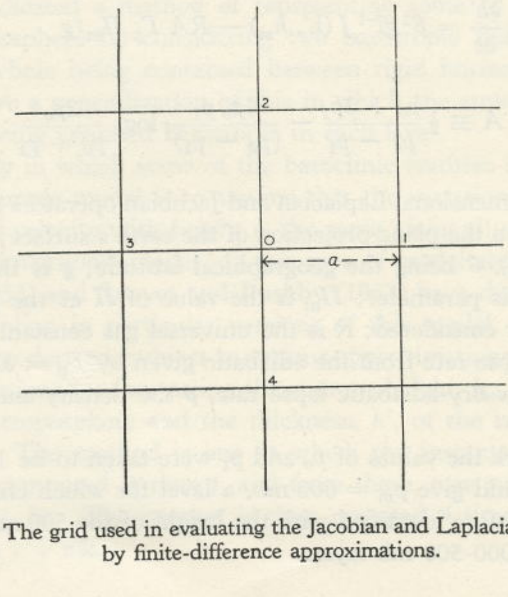


Figure 1. The grid used in evaluating the Jacobian and Laplacian operators by finite-difference approximations.

to obtain the solution of Eq. (2). The 1,000-mb height tendency, $\partial h_0/\partial t$, was calculated by hand from the following relationship :

$$\frac{\partial h_0}{\partial t} = \frac{\partial h_m}{\partial t} - \frac{\partial h'}{\partial t} \quad . \quad . \quad . \quad . \quad (7)$$

LEO, an electronic computing machine owned by Messrs. J. Lyons & Co. Ltd., was used to solve Eqs. (4) and (3) for $\partial h'/\partial t$ and $\Pi_m$. The time taken was about 15 min of which 6 min was spent in reading programme and data, 4 min in printing results and 5 min in computing. A slightly shorter time was taken by the programme which solved Eq. (2) for $\partial h_m/\partial t$.

The field of $\Gamma_p$ was evaluated by using a method similar to that used by Bushby (1952). The computations were then repeated using a constant value of $\Gamma_p$ (38°F per 500 mb), but there was no significant change in the results. The boundary values of $\partial h'/\partial t$ and $\partial h_m/\partial t$ were assumed zero but after the computations had been made an estimate was made of the effect on the answer of using the actual boundary conditions that were observed. The solution of Eq. (2) was repeated using zero values for the vertical advection of vorticity term as it was thought that this term would not have an appreciable effect on the solution. Neglecting this term Eq. (2) becomes

$$\nabla^2 \frac{\partial h_m}{\partial t} + J\,(h_m, \beta^2 gl^{\ 1} \nabla^2 h_m + l) + \tfrac{1}{3} J\,(h', \beta^2 gl^{-1} \nabla^2 h') = 0 \quad . \quad (8)$$

## 4. Results

The three synoptic situations for which computations were made are shown in Figs. 2a, 3a and 4a, and Figs. 2b, 3b and 4b are charts of the computed field of vertical motion, $\Pi_m$. The actual 500-mb height-tendency fields, obtained from the change of 500 mb height in a 12-hr period centred at the times for which the computations were made, are shown in Figs. 2c, 3c and 4c, whilst Figs. 2d, 3d and 4d show the values of the 500-mb height tendency computed by solving Eq. (2) assuming all the boundary values are zero. Similarly, the actual 1,000-500 mb thickness tendency fields, meaned over 12 hr, are shown in Figs. 2e, 3e and 4e, and the thickness tendencies computed from Eq. (4) in Figs. 2f, 3f and 4f. The actual 1,000-mb height-tendency fields, averaged over a 6-hr period centred at the times for which the computations were made, were measured from the surface charts and are shown in Figs. 2g, 3g and 4g. The computed fields of $\partial h_0/\partial t$, assuming zero boundary values, are shown in Figs. 2h, 3h and 4h.

It can be seen from the diagrams that there is a good correspondence between the computed and actual tendency fields. At first, it may seem inconsistent that an equation for $\partial h_m/\partial t$ in which the main terms are those of Charney's equation for a barotropic model atmosphere gives much better results than those found by Charney (1949) or Bushby (1951) when solving the barotropic equation. The additional terms are obviously not responsible for all the improvement, and it therefore seems that the use of a much smaller grid length is an important factor. The 450-mi grid length used in previous work by Charney and Bushby appears to be much too coarse to represent adequately the vorticity field by finite-difference approximations, and Charney *et al.* (1950) came to a similar conclusion.

Correlation coefficients were calculated comparing the computed tendencies with the actual tendencies, and regression coefficients were calculated for the best-fitting straight line giving the actual rate of change ($Y$ feet per hour) in terms of the computed rate of change ($X$ feet per hour) in the form

$$Y = bX + c \quad . \quad . \quad . \quad . \quad . \quad (9)$$

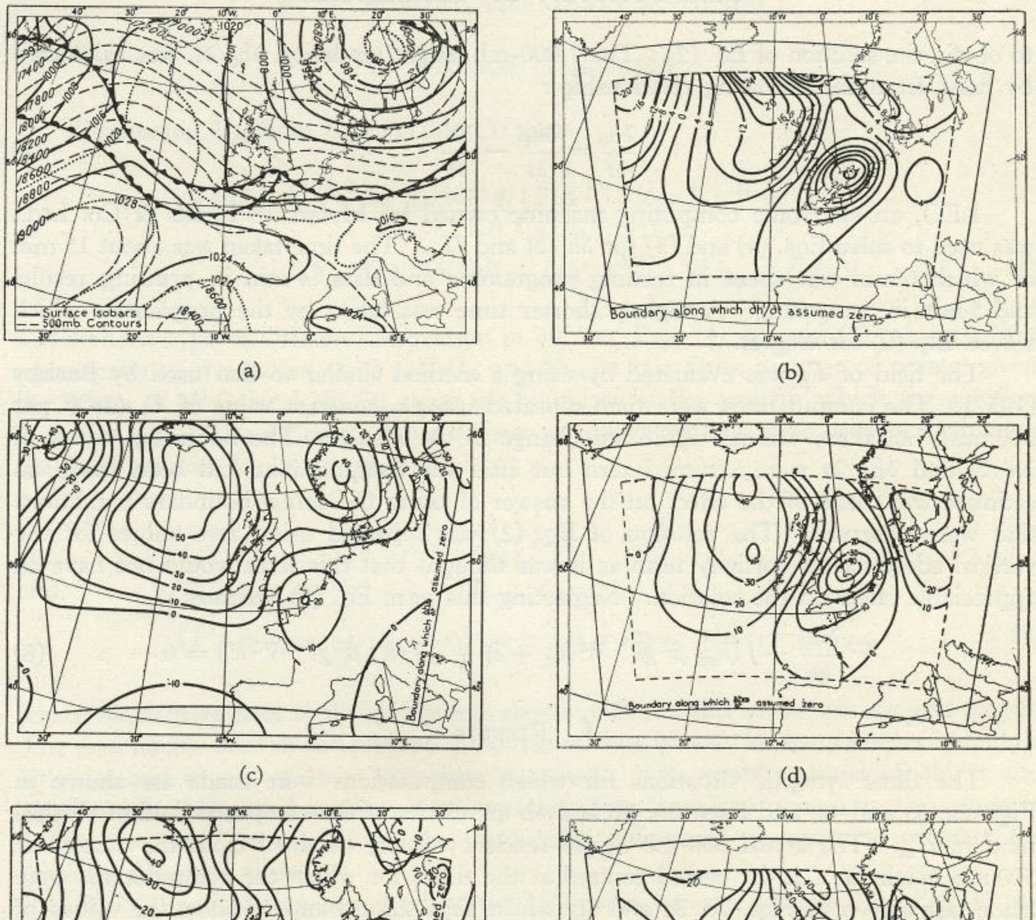


Figure 2. 15 GMT 14 March 1949. (a) Synoptic chart; (b) Vertical motion ($\Pi_m$ in mb/hr); (c) Actual 500 mb height tendency (ft/hr); (d) Computed 500 mb height tendency (ft/hr); (e) Actual 1,000-500 mb thickness tendency (ft/hr); (f) Computed 1,000-500 mb thickness tendency (ft/hr); (g) Actual 1,000 mb height tendency (ft/hr); (h) Computed 1,000 mb height tendency (ft/hr).

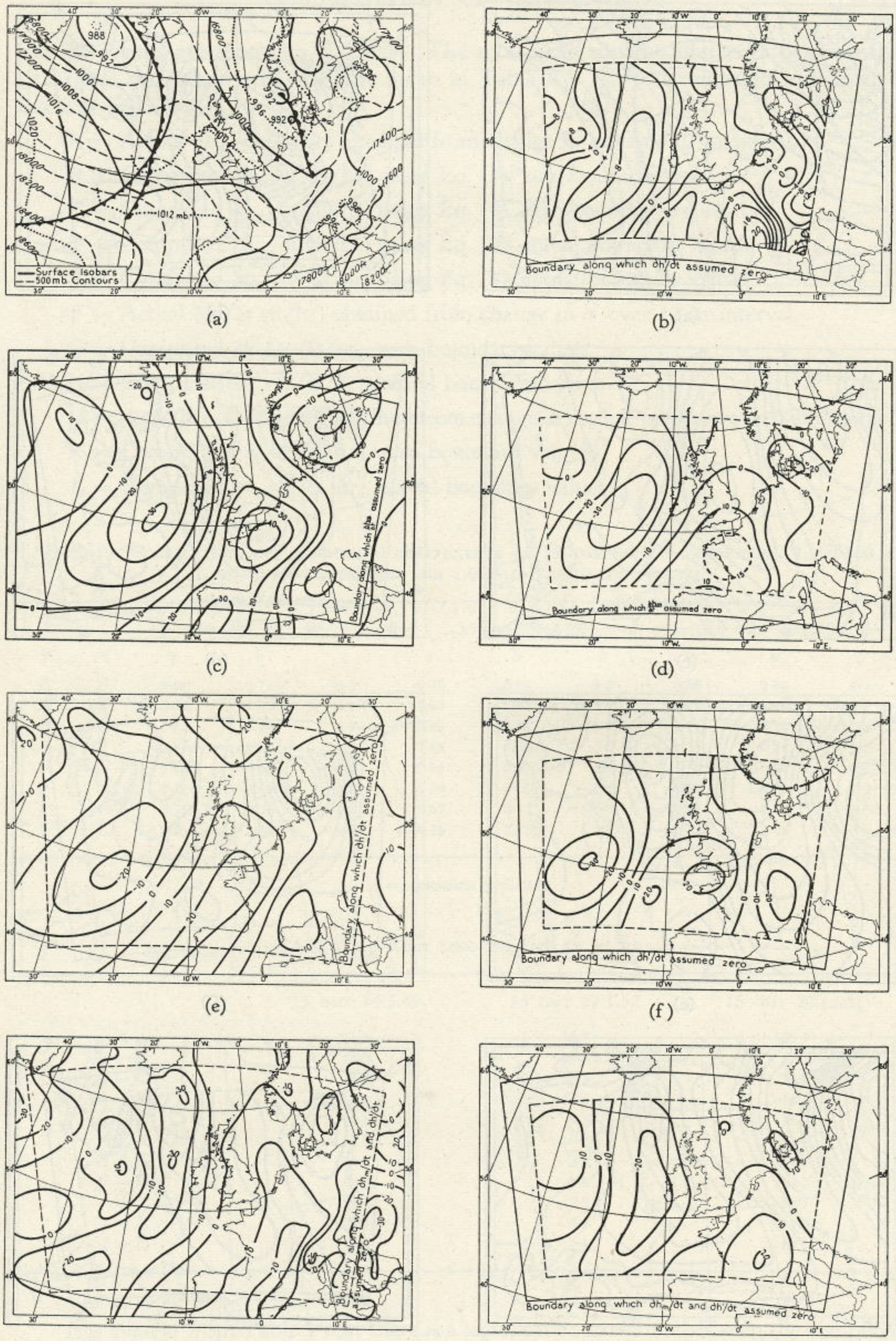


Figure 3. 15 GMT 27 January 1952. (a) Synoptic chart; (b) Vertical motion ($\Pi_m$ in mb/hr); (c) Actual 500 mb height tendency (ft/hr); (d) Computed 500 mb height tendency (ft/hr); (e) Actual 1,000-500 mb thickness tendency (ft/hr); (f) Computed 1,000-500 mb thickness tendency (ft/hr); (g) Actual 1,000 mb height tendency (ft/hr); (h) Computed 1,000 mb height tendency (ft/hr).

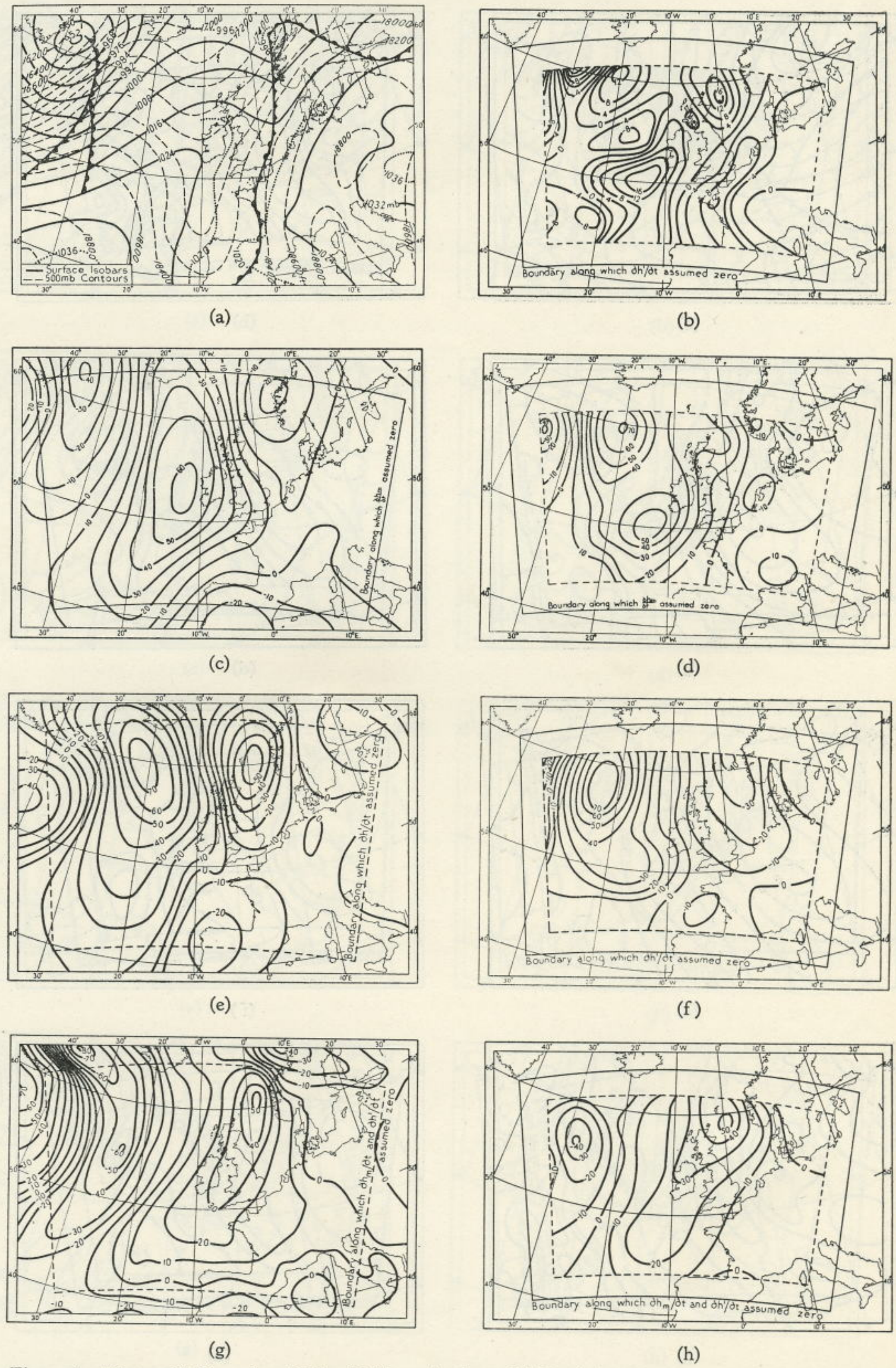


Figure 4. 15 GMT 20 December 1951. (a) Synoptic chart; (b) Vertical motion ($\Pi_m$ in mb/hr); (c) Actual 500 mb height tendency (ft/hr); (d) Computed 500 mb height tendency (ft/hr); (e) Actual 1,000-500 mb thickness tendency (ft/hr); (f) Computed 1,000-500 mb thickness tendency (ft/hr); (g) Actual 1,000 mb height tendency (ft/hr); (h) Computed 1,000 mb height tendency (ft/hr).

and the results are shown in Table 1. The root mean square differences between the actual and computed tendencies are given in Table 2. The notation used in Tables 1 and 2 is as follows :

$A$ = Actual $\partial h_m/\partial t$ (ft/hr) obtained from change in $h_m$ over 12-hr interval.

$B$ = Computed $\partial h_m/\partial t$ (ft/hr) using Eq. (2), zero boundary values.

$C$ = Computed $\partial h_m/\partial t$ (ft/hr) using Eq. (8), zero boundary values.

$D$ = Computed $\partial h_m/\partial t$ (ft/hr) using Eq. (2), actual boundary values.

$E$ = Computed $\partial h_m/\partial t$ (ft/hr) using Eq. (8), actual boundary values.

$F$ = Actual $\partial h'/\partial t$ (ft/hr) obtained from change in $h'$ over 12-hr interval.

$G$ = Computed $\partial h'/\partial t$ (ft/hr), zero boundary values.

$H$ = Computed $\partial h'/\partial t$ (ft/hr), actual boundary values.

$J$ = Actual $\partial h_0/\partial t$ (ft/hr), obtained from change in surface pressure over 6-hr interval.

$K$ = Computed $\partial h_0/\partial t$ (ft/hr), zero boundary values.

$L$ = Computed $\partial h_0/\partial t$ (ft/hr), actual boundary values.

TABLE 1. CORRELATION AND REGRESSION COEFFICIENTS OF 500-MB HEIGHT TENDENCIES, 1,000-500 MB THICKNESS TENDENCIES AND 1,000-MB HEIGHT TENDENCIES.

| | | 15 GMT 14.3.49 | | | 15 GMT 27.1.52 | | | 15 GMT 20.12.51 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $Y$ | $X$ | $r$ | $b$ | $c$ | $r$ | $b$ | $c$ | $r$ | $b$ | $c$ |
| $A$ | $B$ | 0·64 | 0·72 | 6·2 | 0·78 | 0·99 | 9·8 | 0·69 | 0·78 | 7·2 |
| $A$ | $C$ | 0·67 | 0·85 | 3·1 | 0·83 | 1·05 | 5·6 | 0·70 | 0·76 | 4·3 |
| $A$ | $D$ | 0·82 | 0·74 | 4·1 | 0·80 | 0·85 | 6·2 | 0·77 | 0·76 | 7·1 |
| $A$ | $E$ | 0·84 | 0·80 | 1·4 | 0·83 | 0·85 | 2·2 | 0·77 | 0·74 | 4·3 |
| $F$ | $G$ | 0·49 | 0·39 | − 0·4 | 0·76 | 0·67 | 1·5 | 0·85 | 0·94 | 2·4 |
| $F$ | $H$ | 0·72 | 0·49 | 0·2 | 0·80 | 0·61 | − 0·1 | 0·90 | 0·80 | 2·2 |
| $J$ | $K$ | 0·93 | 0·98 | − 0·3 | 0·67 | 0·72 | − 0·9 | 0·83 | 1·18 | − 8·0 |
| $J$ | $L$ | 0·95 | 0·68 | − 0·1 | 0·78 | 0·73 | 2·7 | 0·88 | 1·07 | − 1·3 |

$r$ = correlation coefficient

TABLE 2. ROOT MEAN SQUARES IN FT/HR.

| | 15 GMT 14.3.49 | 15 GMT 27.1.52 | 15 GMT 20.12.51 |
|---|---|---|---|
| $A$ | 21·3 | 15·8 | 26·5 |
| $A-B$ | 16·3 | 14·0 | 17·5 |
| $A-C$ | 14·5 | 10·3 | 16·6 |
| $A-D$ | 12·5 | 12·2 | 16·1 |
| $A-E$ | 11·1 | 9·5 | 15·4 |
| $F$ | 10·5 | 10·6 | 27·3 |
| $F-G$ | 12·3 | 8·9 | 13·7 |
| $F-H$ | 10·8 | 8·3 | 12·7 |
| $J$ | 17·1 | 12·0 | 22·7 |
| $J-K$ | 5·9 | 9·2 | 14·6 |
| $J-H$ | 9·2 | 9·4 | 10·7 |

It is evident from Table 1 that there is a significant correlation between the computed and actual tendency fields, this being greater when the correct boundary values were used. The results are confirmed by Table 2 for two of the situations, but for 15 GMT 14 March 1949 the root mean square of ($F$-$G$) is greater than that of $F$ and the root mean square of ($J$-$K$) is less than that of ($J$-$H$), which may at first seem inconsistent with the correlation coefficients of Table 1. However, this particular situation was one in which a rapidly-moving wave depression, followed by a mobile ridge, was moving ESE across the British Isles and eastern Atlantic and one would therefore not expect the computed instantaneous rates of change to be of the same magnitude as the measured rates of change meaned over a six- or twelve-hour period, although the tendency patterns should correspond. This point emphasizes the fact that it is difficult to obtain a satisfactory objective method of comparing two sets of isopleths of this type.

The values of the three vorticity-advection terms of Eq. (2) were printed by the machine during the course of the computations so that their relative importance could be assessed. In each case the term $J(h_m, \beta^2 gl^{-1} \nabla^2 h_m + l)$, which represents the horizontal advection of the 500 mb absolute vorticity field, is the most important. The term $\frac{1}{3} J(h', \beta^2 gl^{-1} \nabla^2 h')$ adds a contribution at about 60 per cent of the grid points whilst the last term, representing vertical advection of vorticity, makes a contribution at approximately 30 per cent of the grid points. Either of these last two terms can, in certain limited areas, be numerically greater than the first Jacobian. Ignoring the term representing the vertical advection of vorticity when computing $\partial h_m/\partial t$ did not significantly affect the correlation coefficients but decreased the root mean square difference between the actual and computed tendency, although in two of the cases the decrease was only slight. Here again, however, the root mean square differences may give a false impression owing to the use of 12-hr mean tendencies for comparison, and the effect of ignoring the vertical advection of vorticity can only be properly assessed after a time integration has been carried out.

It seems quite obvious from the available information that the assumption of the correct boundary conditions does significantly increase the agreement between the computed and actual tendency fields and it will be necessary to take some notice of this when attempting time integration.

The field of vertical motion computed for 15 GMT 14 March 1949 indicates an area of ascending motion ahead of the mobile wave depression, with a large region of descending motion ahead of the ridge over the eastern Atlantic. Another area of ascending motions is shown ahead of the central Atlantic troughs. In this situation the computed vertical motion agrees quite well with that expected from synoptic experience. The main feature of the computed vertical motion field for 15 GMT 27 January 1952 is the extensive region of subsidence indicated over southern France and the Bay of Biscay. This was associated with a surface-pressure ridge and is on the western side of a cold trough. Other features of interest are the areas of convergence ahead of the frontal troughs in the eastern Atlantic and southern Scandinavia. The vertical-motion field computed for 15 GMT 20 December 1951 is extremely interesting as the twin centres of subsidence to the east of Scotland and to the south-west of Ireland, with an area of convergence over western France, are symptomatic of the cutting-off process and anticyclonic building across the neck of a cold trough. This development actually occurred.

## 5. Conclusion

The main conclusion of the present work is that the Sawyer-Bushby model does give a good representation of the 1,000-500 mb thickness changes, the 500-mb contour-height changes and the field of vertical motion in the three situations to which it has so far been applied. Another, and unexpected result, is that the computed 1,000-mb height tendencies, obtained by subtracting the thickness tendencies from the 500-mb height tendencies, should show such good agreement with actuality.

The effect of boundary conditions seems important if a small area is considered, and before time integration is undertaken it would seem necessary either to increase the area under consideration so that the effect of boundary conditions would not affect the central area, or to make a preliminary estimate of expected changes around the boundary and feed this into the machine.

The present results appear to justify an attempt to produce prognostic charts by the simultaneous time integration of the contour and thickness-tendency equations.

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